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DECOMPOSING k-ARC-STRONG TOURNAMENTS INTO STRONG SPANNING SUBDIGRAPHS

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The so-called Kelly conjecture[1] states that every regular tournament on $2k+1$ vertices has a decomposition into k-arc-disjoint hamiltonian cycles. In this paper we formulate a generalization of that conjecture, namely we conjecture that every k-arc-strong tournament contains k arc-disjoint spanning strong subdigraphs. We prove several results which support the conjecture:

- If $D=(V,A)$ is a 2-arc-strong semicomplete digraph then it contains 2 arc-disjoint spanning strong subdigraphs except for one digraph on 4 vertices.
- Every tournament which has a non-trivial cut (both sides containing at least 2 vertices) with precisely k arcs in one direction contains k arc-disjoint spanning strong subdigraphs. In fact this result holds even for semicomplete digraphs with one exception on 4 vertices.
- Every k-arc-strong tournament with minimum in- and out-degree at least $37k$ contains k arc-disjoint spanning subdigraphs $H_1, H_2, \ldots, H_k$ such that each H_i is strongly connected.

The last result implies that if T is a $74k$-arc-strong tournament with specified not necessarily distinct vertices $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ then T contains $2k$ arc-disjoint branchings $F^-_{u_1}, F^-_{u_2}, \ldots, F^-_{u_k}, F^+_{v_1}, F^+_{v_2}, \ldots, F^+_{v_k}$ where $F^-_{u_i}$ is an in-branching rooted at the vertex u_i and $F^+_{v_i}$ is an out-branching rooted at the vertex v_i, $i=1,2,\ldots,k$. This solves a conjecture of Bang-Jensen and Gutin [3].

We also discuss related problems and conjectures.

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[1] A proof of the Kelly conjecture for large k has been announced by R. Häggkvist at several conferences and in [5] but to this date no proof has been published.

1. Introduction

By a well known theorem of Nash-Williams every $2k$-edge-connected graph contains k edge-disjoint spanning trees (see e.g. [2, Section 9.5]). This implies that every $2k$-edge-connected graph contains k edge-disjoint spanning connected subgraphs. Furthermore, since G contains k edge-disjoint spanning connected subgraphs if and only if it contains k edge-disjoint spanning trees, it can be decided in polynomial time whether a given graph G has such subgraphs $G_1, G_2, \dots, G_k$ (using any polynomial algorithm for finding k edge-disjoint spanning trees). For directed graphs the situation is much more complicated. It is an NP-complete problem to decide whether a given digraph contains arc-disjoint spanning strong subdigraphs (see Corollary 6.7). Furthermore, it is not even known whether there is any degree of strong connectivity which guaranties that a digraph D has two arc-disjoint strong spanning subdigraphs. We will discuss this problem in Section 6.

In this paper we consider the problem above for tournaments. We pose the following conjecture which contains the Kelly conjecture as the special case when $n = 2k+1$.

Conjecture 1.1. A tournament T can be decomposed into k arc-disjoint spanning subdigraphs if and only if T is k-arc-strong.

As mentioned in the abstract we prove several results in this paper which provide some support for the conjecture.

The paper is organized as follows: We first provide the necessary terminology and some basic results that are used later. Then we prove Conjecture 1.1 for the case when the tournament in question contains a non-trivial k-cut (that is a cut $(S, V-S)$ with precisely k arcs from S to $V-S$ and $|S|, |V-S| \geq 2$). In fact we prove this even for semicomplete digraphs with one exception on 4 vertices. In this proof we use two well known non-trivial results from graph theory. In Section 4 we prove that if D is a 2-arc-strong semicomplete digraph, then D contains 2 arc-disjoint spanning strong subdigraphs with one exception on 4 vertices (the same as above). This implies a result by Bang-Jensen on arc-disjoint in- and out-branchings in tournaments [1]. In Section 5 we provide further support for Conjecture 1.1 by showing that every k-arc strong tournament T with minimum in- and out-degree at least $37k$ contains arc-disjoint spanning strong subdigraphs $H_1, H_2, \dots, H_k$ (In particular every $37k$-arc-strong tournament contains k arc-disjoint spanning strong subdigraphs).

It follows from our constructive proof that one can find such a collection $H_1, H_2, \dots, H_k$ of spanning strong subdigraphs in polynomial time. The method we use involves iteratively constructing large arc-disjoint strong subdigraphs of T such that the subdigraph D' induced by the union of the arc sets of these subdigraphs contains a large set of vertices X (containing all vertices from T except possibly a linear function of k vertices) and for any two vertices $x,y \in X$, D' contains k arc-disjoint (x,y)-paths.

Our proof uses several results from [4] on digraphs in which the number of non-neighbours of each vertex is bounded by some constant c and digraphs in which some vertices may have more than c non-neighbours but the total number of such vertices is bounded by some other constant.

Finally, in Section 6 we derive some consequences of our results and discuss related open problems.

2. Terminology and Preliminaries

For notation or terminology not discussed here we refer to [2]. We shall always use the symbol n to denote the number of vertices in the digraph currently under consideration. The digraphs in this paper are finite and have no loops but may have multiple arcs. We use $V(D)$ and $A(D)$ to denote the vertex set and the arc set of a digraph D. The underlying undirected graph of D, denoted $UG(D)$ is the (multi)graph one obtains by suppressing the orientations on each arc. The *complement graph* of an undirected multigraph $G=(V,E)$ is the undirected graph $\bar{G}$ whose vertex set is V and two vertices x,y are joined by an edge in $\bar{G}$ precisely when $xy \notin E$.

The arc from a vertex x to a vertex y will be denoted by xy. If xy is an arc then we say that x *dominates* y and write $x \rightarrow y$ to indicate this. Two vertices x and y are *adjacent* if there is at least one arc between them. Let S be a set of vertices in the digraph D. Then $d^+(S)$ denotes the number of arcs from S to $V(D)-S$ and $d^-(S)$ the number of arcs from $V-S$ to S. In particular, if x is a vertex $d^+(x)$ ($d^-(x)$) denotes the number of arcs whose tail (head) is x. The *degree* $d(x)$ of a vertex x is the number of arcs incident with x that is $d(x)=d^+(x)+d^-(x)$. If we want to specify the degree of a vertex in a subdigraph D' of D we may write $d^+_{D'}(x)$ say. The *minimum degree* of a digraph D, denoted $\delta^0(D)$, is $\delta^0(D)=\min_{x\in V}\{\min\{d^+(x),d^-(x)\}\}$. We use $N^+(S)$ to denote the set of vertices in $V(D)-S$ that are dominated by a vertex in S.

If H is a subgraph of $UG(D)$ then we denote by $d_H(v)$ the degree of v in the undirected subgraph of $UG(D)$ induced by the edges of H. We shall often use this notation when H is the subgraph induced by a subset of $A(D)$ (considered as edges in $UG(D)$).

If $X \subseteq V(D)$ then we denote by $D\langle X\rangle$ the subdigraph *induced* by X in D, that is $D\langle X\rangle$ has vertex set X and contains precisely those arcs from $A(D)$ which have both end vertices in X. A set of vertices S in D is *independent* if no arc of D has both end vertices in S. We denote by $\alpha(D)$ the maximum size of an independent set in D.

By a *cycle (path,* respectively) we mean a directed (simple) cycle (path, respectively). If W is a cycle or a path with two vertices u,v such that u can reach v on W, then $W[u,v]$ denotes the subpath of W from u to v. A cycle (path) of a digraph D is *hamiltonian* if it contains all the vertices of D. A digraph is *hamiltonian* if it has a hamiltonian cycle.

Let U,W be two subsets of $V(D)$. A (U,W)-arc is an arc xy with $x\in U$ and $y\in W$. A (U,W)-path is a path $x_1x_2\dots x_k$ such that $x_1\in U, x_k\in W$ and $x_i\notin U\cup W$ for $i=2,3,\dots,k-1$. An (x,y)-*path* is a path from x to y.

A digraph D is *strongly connected* (or just *strong*) if there exists an (x,y)-path and a (y,x)-path for every choice of distinct vertices x,y of D. A digraph D is k-*arc-strong* for some $k\geq 1$ if $D-A'$ is strong for every subset A' of $A(D)$ such that $|A'|\leq k-1$. We denote by $\lambda(D)$ the maximum k for which D is k-arc-strong. Whenever x and y are distinct vertices of D, we denote by $\lambda_D(x,y)$ the maximum number of arc-disjoint (x,y)-paths. By Menger's theorem $\lambda_D(x,y)\geq k$ for all $x,y\in V(D)$ if and only if D is k-arc-strong.

A digraph D is *semicomplete* if $\alpha(D)=1$. A *tournament* is a semicomplete digraph with no cycles of length 2. The *complete digraph* on n vertices, denoted K_n^* is the digraph in which each pair of distinct vertices induces a 2-cycle.

An *out-branching* (*in-branching*) rooted at r in a digraph D is a spanning tree F in $UG(D)$ which (in D) is oriented in such a way that every vertex except r has precisely one arc coming in to it (going out of it). The following classical result is due to Edmonds:

Theorem 2.1. *[6] Let D be a directed graph and r a vertex of $V(D)$ and let k be a natural number. There exist k arc-disjoint out-branchings (in-branchings) all rooted at r in D if and only if $\lambda_D(r,v)\geq k$ ($\lambda_D(v,r)\geq k$) for every $v\in V(D)-r$.*

Lemma 2.2. *Let D be a semicomplete digraph and let k be an integer, such that $\sum_{x\in V(D)}\max\{0,k-d^+(x)\}\leq k-1$. Then we must have $|V(D)|\geq k+1$.*

Proof. Let $n=|V(D)|$ and note that the following must hold $n(n-1)\geq |A(D)|=\sum_{x\in V(D)}d^+(x)\geq nk-(k-1)$. By rearranging the terms we get that $(n-1)(n-k)\geq 1$, implying that $n\geq k+1$. ∎

Lemma 2.3. *Every 2-arc-strong semicomplete digraph H has 3 distinct vertices q_1,q_2,q_3 such that $H-q_i$ is strong for $i=1,2,3$.*

Proof. Since strong semicomplete digraphs are vertex pancyclic, it is easy to see that every strong semicomplete digraph D on at least 4 vertices has two vertices x_1, x_2 such that $D - x_i$ is strong for $i = 1, 2$. If H has at most 5 vertices then it is easy to check that $H - x$ is strong for every vertex x so we may assume that $|V(H)| \geq 6$. Let x_1, x_2 be chosen such that $H - \{x_1, x_2\}$ is strong. Since $\delta^0(H) \geq 2$ it follows that $H - x_i$ is strong for $i = 1, 2$. Let $H' = H - \{x_1, x_2\}$ and let x_3, x_4 be chosen such that $H' - x_i$ is strong for each $i = 3, 4$. Now it is easy to show that $H - x_3$ is strong unless all arcs between $\{x_1, x_2\}$ and $V - \{x_1, x_2, x_3\}$ have the same direction. In that case $H - x_4$ is strong. ∎

Lemma 2.4. *Let D be a k-arc-strong semicomplete digraph and let $x \in V(D)$ have $d^+(x) = k$ and $d^-(x) > k$. If there exists a 2-cycle xyx in D, such that $d^+(y) > k$ then $D - yx$ is k-arc-strong.*

Proof. Let $D' = D - yx$ and assume that D' is not k-arc-strong. Let $\emptyset \neq S \subset V(D')$ be defined such that $d^+(S)$ is minimum. By our assumption we get that $d^+(S) < k$. Note that as D is k-arc-strong, we must have $y \in S$ and $x \notin S$. Let $D^* = D\langle S\rangle$ and note that $\sum_{u \in V(D^*)} \max\{0, k - d^+_{D^*}(u)\} \leq k-1$. By Lemma 2.2 it follows that $|S| \geq k+1$. If there are r arcs from S to x in D', then there must be at least $k+1-r$ arcs from x to S (as D' is semicomplete), and at most $k - (k+1-r) = r-1$ arcs from x to $V(D') - S$ ($S \cup x \neq V$, as $d^-_D(x) > k$). Therefore $d^+_{D'}(S \cup x) < d^+_{D'}(S)$, a contradiction. ∎

Lemma 2.5. *Let D be a k-arc-strong digraph, and let C be a cycle in D. Then the digraph obtained by reversing all arcs in the cycle C is also k-arc-strong.*

Proof. Let D' be the digraph obtained by reversing all arcs in the cycle C. Let $\emptyset \neq S \subset V(D)$ be arbitrary. Note that the cycle C has equally many arcs from S to $V(D) - S$ as it does from $V(D) - S$ to S. Therefore $d^+_{D'}(S) = d^+_D(S)$, which implies the lemma. ∎

Lemma 2.6. *Let D be a k-arc-strong semicomplete digraph, and let $x \in V(D)$, have the property that $d^+(w) = k$, for all $w \in N^+(x)$. If $\delta^0(D - x) \geq k$, then $D - x$ is k-arc-strong.*

Proof. Let $D' = D - x$ and assume that D' is not k-arc-strong. Let $\emptyset \neq S \subset V(D')$ be defined such that $d^+_{D'}(S)$ is minimum. By our assumption we get that $d^+_{D'}(S) < k$. As D is k-arc-strong, there must be a vertex $w \in N^+(x)$, which also belongs to $V(D') - S$ (if not then $d^+(S \cup \{x\}) < k$). By Lemma 2.2 we get that $|S| \geq k+1$. Analogously to the proof of Lemma 2.4 we can obtain a contradiction to the minimality of $d^+(S)$ (we consider $S \cup \{w\}$). ∎

3. Decomposing a semicomplete digraph D with a non-trivial $\lambda(D)$-cut

We shall use two well known results in this section. The first one was known under the name ''The Evans conjecture" and originally dealt with partially completed Latin squares, but it is easily restated as below.

Theorem 3.1. *[8] Let B be a complete bipartite graph (undirected), with n vertices in each partite set, and let R be a set of edges in B, such that $|R| \leq n-1$. Then we can decompose $E(B)$ into n edge-disjoint matchings $M_1, M_2, \ldots, M_n$, such that $|M_i \cap R| \leq 1$ for all $i = 1, 2, \ldots, n$.*

Corollary 3.2. *Let $B = (X, Y, E)$ be a complete bipartite graph (undirected), with $|X| = t$, $|Y| = s$ and $t > s$. Let R be a set of edges in B, such that $|R| \leq s$. Then we can colour the edges of B by $|R|$ colours in such a way that all edges in R receive distinct colours and every vertex in $X \cup Y$ is incident with all $|R|$ colours.*

Proof. Add $t-s$ new vertices to Y and join them completely to X. Let $B' = (X, Y', E')$ denote the resulting complete bipartite graph with $2t$ vertices. By Theorem 3.1 we can colour the edges of B' by t colours in such a way that all edges in R receive different colours and every vertex in $X \cup Y'$ is incident with all t colours. If some vertex $x \in X$ has its edge of colour $i \leq |R|$ to a vertex in $y' \in Y' - Y$ then it must have an edge of some colour $j > |R|$ to a vertex $y \in Y$. Now recolour the edge xy by i. It is easy to see that we can continue doing so until every vertex in X has edges of each colour $1, 2, \ldots, |R|$ to vertices in Y. Since we only recolour an edge incident with $y \in Y$ if it has a colour $j > |R|$, every vertex in Y has edges of each colour $1, 2, \ldots, |R|$ to vertices in X when the process stops. Now delete the vertices of $Y' - Y$ and recolour any remaining edge of colour $j > |R|$ by colour 1 and the claim is proved. ∎

The second result we use is due to Tillson and characterizes when one can decompose the arc set of the complete digraph into arc-disjoint hamiltonian cycles.

Theorem 3.3. *[11] The arc set of the complete digraph on k vertices can be decomposed into arc-disjoint hamiltonian cycles if and only if $k \neq 4, 6$.*

Let S_{2k} be the semicomplete digraph one obtains from two disjoint copies of K_k^* by adding a perfect matching from one copy to the other and all remaining arcs in the opposite direction (see Figure 1).

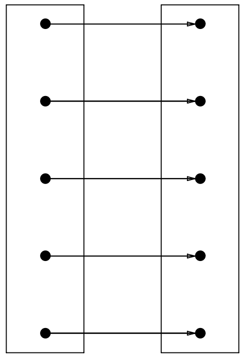

Figure 1. The semicomplete digraph S_{2k}. Each box is a K_k^* and all arcs go from the right box to the left except for the k arcs shown which form a perfect matching. Note that there is no 2-cycle with one end in each K_k^*.

Lemma 3.4. *Let D be a semicomplete digraph which is isomorphic to S_{2k} for some $k \geq 2$. Then D contains k-arc-disjoint spanning strong subdigraphs except when $k = 2$.*

Proof. It is easy to see that S_4 is 2-arc-strong but has no two arc-disjoint spanning strong subdigraphs. Figures 2 and 3 show decompositions of S_6 and S_{10} into 3 and 5 arc-disjoint spanning strong subdigraphs respectively. Hence we may assume that $k \notin \{2,3,5\}$. By Theorem 3.3 the complete digraph K_{k+1}^* has a decomposition into k arc-disjoint hamiltonian cycles. Thus by deleting the $k+1$'st vertex we obtain a decomposition of K_k^* into k arc-disjoint hamiltonian paths $P_1, P_2, \dots, P_k$ such that P_i starts in x_i and ends in $x_{\pi(i)}$ for $i = 1,2,\dots,k$ where the vertex set of K_k^* is $\{x_1, x_2, \dots, x_k\}$ and π is a permutation of $\{1,2,\dots,k\}$. Note that, by the construction of the paths $P_1, P_2, \dots, P_k$, $\pi(i) \neq i$ holds for all i. Now let the vertices of S_{2k} be $\{x_1, x_2, \dots, x_k\} \cup \{x'_1, x'_2, \dots, x'_k\}$, where the labelling of the last k vertices is chosen such that $\{x_i x'_i : i = 1,2,\dots,k\}$ are the only arcs from $\{x_1, x_2, \dots, x_k\}$ to $\{x'_1, x'_2, \dots, x'_k\}$. Let P'_i be the $(x'_{\pi(i)}, x'_i)$-path in the right copy of K_k^* corresponding to the path P_i in the left copy of K_k^* with all arcs reversed. Let $A(H_i) = A(P_i) \cup A(P'_i) \cup \{x'_i x_{\pi(i)}, x'_{\pi(i)} x_i, x_{\pi(i)} x'_{\pi(i)}\}$ for $i = 1,2,\dots,k$. Then $H_1, H_2, \dots, H_k$ are the desired spanning strong subdigraphs. ∎

Theorem 3.5. *Let $k \geq 1$ and let D be a k-arc-strong semicomplete digraph such that there a set $S \subset V(D)$, with $2 \leq |S| \leq |V(D)| - 2$ and $d^+(S) = k$. There exist k arc-disjoint strong spanning subgraphs of D except if $D = S_4$.*

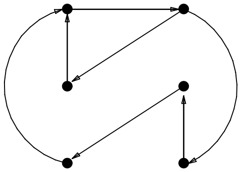

Figure 2. Decomposing the semicomplete digraph S_6 into 3 arc-disjoint spanning strong subdigraphs.

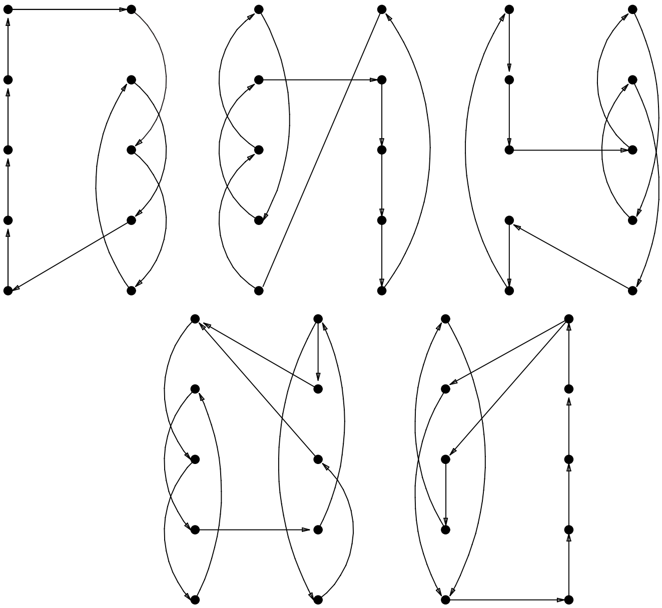

Figure 3. Decomposing the semicomplete digraph S_{10} into 5 arc-disjoint spanning strong subdigraphs.

Proof. We may assume that D is not isomorphic to S_4 since we saw in the beginning of the proof of Lemma 3.4 that S_4 has no two arc-disjoint spanning strong subdigraphs.

Using an argument analogous to that in the proof of Lemma 2.2, we obtain that $k \le |S| \le n-k$ (by showing that $|S| \ge k$ and $|V(D)-S| \ge k$, respectively). If $|S| = |V-S| = 2$ then D contains S_4 as a proper spanning subdigraph and it is easy to check that adding any arc to S_4 will result in a digraph with two arc-disjoint strong spanning subdigraphs. Hence we may assume that $n \ge 5$. Let $e_1, e_2, \dots, e_k$ be the k arcs from S to $V(D)-S$, and let $e_i = x_i y_i$, for $i = 1, 2, \dots, k$. Let $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$. Note that we may have $|X| < k$ or $|Y| < k$ or both. We may assume, by reversing all arcs if necessary, that $|V-S| \ge |S|$.

By Lemma 3.4 and the remark above, we may assume that $|V-S|>|S|$ if $|S|=k$. By Corollary 3.2 (with $R=\{e_1,e_2,\dots,e_k\}$) we can colour all arcs between S and $V(D)-S$ with k colours, such that the arcs from S to $V(D)-S$ get different colours and every vertex in V is incident with arcs of all k colours. Note that if $|V-S|=|S|>k$ this follows from Theorem 3.1.

Assume, without loss of generality, that the arc x_iy_i is coloured with colour i, and let F_i contain all arcs between S and $V(D)-S$ of colour i.

By Theorem 2.1 there exists k arc-disjoint out-branchings $U_1,U_2,\dots,U_k$, in $D\langle V(D)-S\rangle$, such that U_i is rooted at y_i, for $i=1,2,\dots,k$ (consider k arc-disjoint out-branchings from any vertex in S. Each of these must contain exactly one of the arcs $e_1,e_2,\dots,e_k$. Thus the out-branching that contains the arc e_i must contain an out-branching from y_i in $D\langle V(D)-S\rangle$). Analogously, there exists k arc-disjoint in-branchings $V_1,V_2,\dots,V_k$, in $D\langle S\rangle$, such that V_i is rooted at x_i, for $i=1,2,\dots,k$. Let $T_i=V_i\cup U_i\cup F_i$, for $i=1,2,\dots,k$. Clearly $T_1,T_2,\dots,T_k$ are arc-disjoint and spanning. Each T_i is furthermore strong: by the construction of the colouring, every vertex in V is incident to an arc of colour i, every vertex in $V(D)-S-y_i$, has an arc in T_i into S, and hence every vertex in V can reach y_i (via V_i and the arc x_iy_i) and every vertex in $S-x_i$ has an arc in T_i from $V(U_i)$, implying that all vertices of V can be reached by y_i. This completes the proof. ∎

4. Decomposing 2-arc-strong semicomplete digraphs

In this section we solve completely the problem of decomposing a 2-arc-strong semicomplete digraph into two strong spanning subdigraphs.

Theorem 4.1. *Let D be a 2-arc-strong semicomplete digraph, on n vertices. Then D has two arc-disjoint spanning strong subgraphs, if and only if it is not isomorphic to S_4, defined in Figure 1.*

Proof. Let D be a 2-arc-strong semicomplete digraph. We will now prove the theorem by induction on $n=|V(D)|$. If $n=3$ then $D=K_3^*$ which has two arc-disjoint 3-cycles.

Suppose first that $n=4$. As we argued earlier S_4 does not have two arc-disjoint spanning subdigraphs. If D is not isomorphic to S_4, then let C be any 4-cycle in D, and note that if $D-A(C)$ is not strong, then D contains S_4 as a subgraph. It is not difficult to check that if we add any additional arc to S_4, then the new digraph will have the desired property.

If $n=5$, then let C be any 5-cycle in D. If $D-A(C)$ is not strong, then there are two strong components Q_1 and Q_2 in D–$A(C)$ such that there is no arc from Q_2 to Q_1. However it is not difficult to check that Q_1 and Q_2 form a non-trivial cut in D (with $d^+_D(Q_2)=2$), so we are done by Theorem 3.5.

So now assume that $n \geq 6$. Let $Q=\{q|d^+(q)=2\}$ and $W=\{w|d^-(w)=2\}$. Furthermore assume that there is no set S, such that $2 \leq |S| \leq n-2$, and $d^+(S)=2$, as then we would be done by Theorem 3.5. For any vertex w with the property that $D-w$ is strong we let the set $R(w)$ be defined by $R(w)=\{x \in V(D)-w : \min\{d^+_{D-w}(x), d^-_{D-w}(x)\}=1\}$. Without loss of generality assume that $|Q| \geq |W|$ (otherwise reverse all arcs), and consider the following cases.

1. $|Q| \leq 2$: Suppose first that there exists a vertex w such that $D'=D-w$ is strong and $R(w)=\emptyset$ or equivalently $\delta^0(D') \geq 2$. Let $V(D')=\{x_1,x_2,\ldots,x_{n-1}\}$, where the labelling is chosen such that $d^+_{D'}(x_1) \geq d^+_{D'}(x_2) \geq \ldots \geq d^+_{D'}(x_{n-1})$. Let a be maximum, such that $x_a \rightarrow w$. If there is any vertex z such that $w \rightarrow z$ and x_a does not dominate z then we choose b as small as possible such that $w \rightarrow x_b$ and x_a does not dominate x_b. If such a b exists, then let $D''=D' \cup x_a x_b$, otherwise let $D''=D'$.
   We will show that D'' is 2-arc-strong (it is clearly strong). Suppose that Z is a proper subset of $V(D'')$, such that $d^+_{D''}(Z)=1$, let $\bar{Z}=V(D)-Z$, and let xy be the arc going from Z to $\bar{Z}$. We now consider D' instead of D''. As D' is strong we also have $d^+_{D'}(Z)=1$. Note that $|Z|,|\bar{Z}|>1$ as $\delta^0(D') \geq 2$. Furthermore, as D' is strong there is at least one arc from y to a vertex in $\bar{Z}$ and at least one arc into x from a vertex in Z. Thus we have $d^+_{D'}(q) \geq |Z|$, for all $q \in \bar{Z}$ and $d^+_{D'}(q) < |Z|$, for all $q \in Z$. As D has no non-trivial 2-cut there are least two vertices in Z dominating w and at least two vertices in $\bar{Z}$ which are dominated by w. So $x_a \in Z$ and $x_b \in \bar{Z}$ (and b exists), contradicting the assumption that $d^+_{D''}(Z)=1$. Hence we have shown that D'' is a 2-arc-strong semicomplete digraph.
   By induction we can find two arc-disjoint strong spanning subgraphs H_1, H_2 in D''. If one of these, say w.l.o.g., H_1 uses the arc $x_a x_b$ (defined above), then replace this arc by the path $x_a w x_b$ and include w in H_2 using arbitrary arcs uw and wv, where $u \neq x_a, v \neq x_b$. Otherwise include w in each H_i using two distinct arcs into w and two distinct arcs leaving w. In both cases we obtain two arc-disjoint strong spanning subgraphs in D.
   Hence we may assume that $|R(w)| \geq 1$ for every w such that $D-w$ is strong. If there exists a w such that $D-w$ is strong and $|R(w)|=1$, then w.l.o.g. (by reversing all arcs if necessary) $R(w)=\{x\}$ where $N^+_D(x)=\{y,w\}$ for some y. Now it is not difficult to see that we can argue as we did above (If there is a Z with $d^+_{D'}(Z)=1$ and $|Z|,|\bar{Z}|>1$ then we can use the same argument and if $Z=\{x\}$ then x_a will become x and x_b exists since D has no non-trivial 2-cut).

Now consider the case when $|R(w)| \geq 2$ for every w such that $D-w$ is strong. By Lemma 2.3, D has at least 3 vertices w_1, w_2, w_3 such that $D-w_i$ is strong. Now it follows from the fact that $|W| \leq |Q| \leq 2$ that we can choose w such that $D-w$ is strong and precisely one vertex y in $D-w$ has in-degree 1 and precisely one vertex x has out-degree 1 in $D-w$ (and since we were not in any of the cases above, D has at most 4 vertices z such that $D-z$ is strong). If there is no arc from x to y, then it follows from the arguments above that $D-w+xy$ is 2-arc-strong and we can finish as above. Hence we may assume that $x \rightarrow \{w,y\}$ and $w \rightarrow y$. Let $M = D - \{x,y,w\}$ and let $M_1, M_2, \ldots, M_r$ be the acyclic ordering of the strong components of M (i.e. there is no arc from M_j to M_i for any i,j such that $1 \leq i < j \leq r$).

If $r=1$, then let $C = u_1u_2\ldots u_{n-3}u_1$ be a hamiltonian cycle of M which is chosen such that wu_1 and $u_{n-3}w$ are arcs of D (this is possible since w has an in-neighbour and an out-neighbour on C). Let H_1 be the hamiltonian cycle $xwyC[u_1,u_{n-3}]x$ and let $A(H_2) = \{yu_i : i \neq 1\} \cup \{u_jx : j \neq n-3\} \cup \{wu_1, u_{n-3}w, xy\}$. Now H_1 and H_2 are the desired spanning subdigraphs. Hence we may assume that $r \geq 2$. Observe that if $M_1 = \{z\}$ then $w \rightarrow z$ (since x has no arc to M) and similarly, if $M_r = \{z'\}$ then $z' \rightarrow w$. Now it is easy to see that if $|M_1| = |M_r| = 1$ then D has a hamiltonian cycle C' so that $D - C'$ is strong. Hence we may assume that at least one of M_1, M_r has size greater than one. If both have size at least 2 then let uv be any arc in M_1 and pq any arc in M_r and let P be a hamiltonian path in M from v to p. Then $D - C''$ is strong where C'' is the hamiltonian cycle $yPxwy$. To see this it suffices to note that $D - C''$ contains the arcs $\{y \rightarrow z : z \in M - v\} \cup \{z \rightarrow x : z \in M - p\} \cup \{uv, pq, xy, wf, gw\}$, where f and g are an out-neighbour and an in-neighbour, respectively, of w in M. In the remaining case we may assume w.l.o.g. that $M_1 = \{z\}$ and $|M_r| \geq 2$. Let pq any arc in M_r and let P' be a hamiltonian path in M from z to p. Then $C^* = xwyP'x$ is a hamiltonian cycle and again we see that $D - C^*$ is strong unless z is the only in-neighbour of w in M. In this last case let H be the cycle $xwP'x$ and let z' be the third vertex on P'. Let $A(H_1) = A(H) \cup \{yz', wy\}$ and let $A(H_2) = \{yv : v \in M - z'\} \cup \{vx : v \in M - p\} \cup \{z \rightarrow z', pq, zw, wp, xy\}$. Then H_1 and H_2 are the desired subdigraphs.

2. $|Q| = 3$ and Q is acyclic: Then let $Q = \{q_1, q_2, q_3\}$, such that $A(D\langle Q \rangle) = \{q_1q_2, q_1q_3, q_2q_3\}$. Note that in $D' = D - \{q_1\}$ we have $\delta^0(D') \geq 2$. By Lemma 2.6, D' is 2-arc-strong. By induction there exists two arc-disjoint spanning strong subgraphs, T_1 and T_2, in D'. Since q_1 is dominated by all vertices of $D-Q$ and dominates q_2, q_3 we can add q_1 to each of T_1, T_2 and obtain the desired subdigraphs of D.

3. $|Q|=3$ and Q contains a cycle: Then as Q is not a non-trivial cut (with $d^+(Q)=2$), we must have that Q is an induced 3-cycle, say $q_1q_2q_3q_1$. We now consider the cases when $|N^+(Q)|=1$ and when $|N^+(Q)|>1$, separately.
   If $|N^+(Q)|=1$, then let $N^+(Q)=\{w\}$ and let U_1 and U_2 be two arc-disjoint out-branchings rooted at w in $D-Q$. Such branchings exist, by Theorem 2.1, since every out-branching rooted in w in D must contain an out-branching from w in $D\langle V-Q\rangle$ as $N^+(Q)=\{w\}$. Let $T_1=U_1\cup\{q_1q_2,q_2q_3,q_3w\}\cup\{sq_1|s\in V(D)-Q-\{w\}\}$ and let $T_2=U_2\cup\{q_3q_1,q_1w,q_2w\}\cup\{sq_2,sq_3|s\in V(D)-Q-\{w\}\}$. Note that T_1 and T_2 are arc-disjoint spanning strong subgraphs in D.
   If $|N^+(Q)|>1$, then let $N^+(Q)=\{w_1,\dots,w_r\}$, where $r=|N^+(Q)|$. As there are only 3 arcs from Q to $N^+(Q)$, there is a vertex in $N^+(Q)$, which only has one arc into it from Q. Without loss of generality assume that this is w_1, and assume that the arc from Q into w_1 is q_3w_1. Note that $q_3w_1q_2q_3$ is a 3-cycle in D. Let D' be the semicomplete digraph obtained from D by reversing the arcs in the cycle $q_3w_1q_2q_3$. By Lemma 2.5 D' is 2-arc-strong. The set $\{q|d^+_{D'}(q)=2\}$ is now acyclic. As above (using Lemma 2.6), we see that there exists two arc-disjoint spanning strong subgraphs, T_1 and T_2, in $D'-\{q_3\}$. If none of T_1,T_2 use the arc q_2w_1, then we can add q_3 to each of these and obtain the desired subdigraphs of D. Otherwise we may assume w.l.o.g. that T_1 uses q_2w_1. Now replace that arc by the arcs q_2q_3,q_3w_1 and insert q_3 in T_2 using the arc q_3q_1 and any arc into q_3 from $D-\{q_1,q_2,q_3,w_1\}$. This gives us the desired subdigraphs in D.
4. $|Q|\geq 4$: Then it is not difficult to check that $|Q|=4$ and Q is a non-trivial cut (with $d^+(Q)=2$), a contradiction.

As the above cases exhaust all possibilities, we have proved the theorem. ∎

Corollary 4.2. *[1] Let D be a 2-arc-strong semicomplete digraph and let u,v be arbitrary vertices of D. Then D contains arc-disjoint branchings F_u^+,F_v^- such that F_u^+ is an out-branching from u and F_v^- is an in-branching into v.*

Proof. This follows immediately from Theorem 4.1 except in the case when $D=S_4$ in which case the claim is easily verified. ∎

5. Decomposing tournaments with high minimum degree

In this section we shall use the following results which were first proved in [4] (as Theorem 4.2 and Corollary 4.3).

Theorem 5.1. *Let D be a strong digraph, let R denote the complement graph of $UG(D)$ and let $c \geq 0$ be an integer. Suppose we have $\sum_{d_R(u)>c}[d_R(u)-c] \leq q$. Then there exists a strong spanning subdigraph H of D such that $|A(H)| \leq n+c+\sqrt{2q}$.*

Corollary 5.2. *Suppose $D=(V,A)$ satisfies the hypothesis of Theorem 5.1 and let $\tilde{D}$ be obtained from D by subdividing some arcs. Then $\tilde{D}$ contains a strong subdigraph $\tilde{H}=(\tilde{V},\tilde{A})$ such that $V \subseteq \tilde{V}$ and $|\tilde{A}| \leq |\tilde{V}|+c+\sqrt{2q}$.* ∎

The following lemma was also proved in [4].

Lemma 5.3. *Let T be a tournament and R a subset of $A(T)$. Suppose that $\sum_{\{u \in V(T): d_R(u)>c\}}[d_R(u)-c] \leq q$ and that z is a vertex such that*

$$d_T^+(u) \leq d_T^+(z)+\gamma \text{ for all } u \in V(T). \tag{1}$$

Let W be the set of vertices which are not reachable from z by a directed path in $D=T-R$. Then

$$|W| \leq 2c+2d_R(z)+2\gamma-1+\sqrt{2q}.$$

We will now prove the following theorem which implies that we can always obtain about $\frac{1}{37}\lambda(D)$ arc-disjoint spanning strong subdigraphs in any tournament. Note that in the case when $\lambda(D)<37k$ the result below follows from Theorem 3.5.

Theorem 5.4. *Let T be a k-arc-strong tournament, with $\delta^0(T) \geq 37k$. Then there exists k arc-disjoint spanning strong subgraphs in T.*

Proof. Let $T=(V,A)$ be a k-arc-strong tournament on n vertices, with $\delta^0(T) \geq 37k$. Let $v_1,v_2,\ldots,v_n$ be an ordering of the vertices of T such that $d^+(v_1) \leq d^+(v_2) \leq \ldots \leq d^+(v_n)$. Note that since T is a tournament this ordering also satisfies $d^-(v_1) \geq d^-(v_2) \geq \ldots \geq d^-(v_n)$. Let $X=\{v_{n-k+1},v_{n-k+2},\ldots,v_n\}$ and $Y=\{v_1,v_2,\ldots,v_k\}$.

Since T is k-arc-strong, it follows from a slight extension of Menger's theorem (see e.g. [2, Exercise 7.17]) that there are k arc-disjoint paths $P_1,P_2,\ldots,P_k$ from Y to X such that all end vertices of these paths are disjoint. Let $y_1,y_2,\ldots,y_k$ and $x_1,x_2,\ldots,x_k$ be distinct vertices which are chosen such that $X=\{x_1,x_2,\ldots,x_k\}$, $Y=\{y_1,y_2,\ldots,y_k\}$ and P_i is a (y_i,x_i)-path for $i=1,2,\ldots,k$. Note that P_i may contain several vertices from $X \cup Y-\{x_i,y_i\}$.

Define $c_i=\gamma_i=2k-2$, $i=1,2,\ldots,k$ and $q_1,q_2,\ldots,q_{k+1}$ recursively as follows:

$$\begin{aligned} & q_1 = 0, \\ & q_i = q_{i-1}+2(2k-2+\sqrt{2q_{i-1}}), \quad i=2,3,\ldots,k+1. \end{aligned} \tag{2}$$

Note that we have $q_1 < q_2 < \ldots < q_{k+1}$ and it is not difficult to show by induction on i that $q_i < 16k(i-1)$. In particular, we have $q_{k+1} < 16k^2$.

We will now construct arc-disjoint strong subdigraphs $H_1, H_2, \ldots, H_k$ (in that order) of T. Given $H_1, H_2, \ldots, H_{i-1}$ $(i = 1, 2, \ldots, k-1)$, we define the following sets

- $L_i = A(H_1) \cup A(H_2) \cup \ldots \cup A(H_{i-1})$.
- $R_i = L_i \cup A(P_{i+1}) \cup A(P_{i+2}) \cup \ldots \cup A(P_k)$.
- $Z_i = \{z \in V | d_{L_i}(z) \geq 10k\}$.
- $W_i = X \cup Y \cup Z_i - \{x_i, y_i\}$.

Define the digraphs D_i, D_i^* and D_i^{**} as follows: $D_i = T - R_i$ and let $D_i^* = D_i - W_i$. Finally let $D_i^{**} = D_i^* \cup V(P_i) \cup A(P_i)$.

Assume that we have found arc-disjoint strong subdigraphs $H_1, H_2, \ldots, H_{i-1}$ so that the following holds.

(A): $\sum_{j=1}^{i-1} \sum_{d_{H_j}(u) > 2} [d_{H_j}(u) - 2] \leq q_i$.
(B): $\sum_{d_{R_i}(u) > c_i} [d_{R_i}(u) - c_i] \leq q_i$.
(C): $|Z_i| \leq 2k$ and $|W_i| \leq 4k - 2$.
(D): $d_{R_i}(x_i), d_{R_i}(y_i) \leq 2k - 2$.

We claim that D_i^{**} is strongly connected. Let Q be all the vertices in D_i^* which cannot be reached by x_i. By Lemma 5.3 (with $\gamma = 4k - 2$) we get that

$$\begin{aligned} |Q| &\leq 2c_i + 2d_{R_i}(x_i) + 2(4k-2) - 1 + \sqrt{2q_i} \\ &\leq (4k-4) + (4k-4) + (8k-4) - 1 + 6k \\ &= 22k - 9. \end{aligned} \tag{3}$$

Note that D_i^* contains no vertex from Z_i (neither x_i nor y_i belongs to Z_i by (D)). Let $r \in V(D_i^*)$ be arbitrary. Since $r \notin Z_i$ we have

$$\begin{aligned} d^-_{D_i^*}(r) &\geq 37k - 10k - k - (4k-2) \\ &= 22k + 2. \end{aligned} \tag{4}$$

This follows from the fact that there are at least 37k arcs into r in T and as we have used at most $10k$ of them in the H_j's so far (as $r \notin Z_i$) at most k in the P_j's and at most $4k - 2$ into vertices in W_i (by (C)). It follows from (3) and (4) that r can be reached from x_i (as if r couldn't be reached then $N^-(r)$ couldn't be reached). Analogously we get that all vertices in D_i^* can reach y_i in D_i^*. Since $D_i^{**} = D_i^* \cup V(P_i) \cup A(P_i)$ it follows that D_i^{**} is strong, which proves the claim.

By Corollary 5.2 (with c_i and q_i, in the place of c and q in the theorem), we can find a strong spanning subdigraph H_i of D_i^{**} with $|A(H_i)| \leq |V(H_i)| + c_i + \sqrt{2q_i}$.

We can now prove (A)–(D) by induction on i. Suppose first that $i=1$. Then (A) holds vacuously and since $L_1=\emptyset$ and $R_1=A(P_2)\cup\ldots\cup A(P_k)$ it follows that (C) and (D) hold. Finally, as no vertex is incident to more than two arcs on each P_j, $j=1,2,\ldots,k$ and $c_1=2k-2$, (B) also holds.

Suppose now that (A) and (B) holds for some $i<k$. We will now show that (A) and (B) holds for $i+1$. By the construction of H_i above we have $|A(H_i)|\le|V(H_i)|+c_i+\sqrt{2q_i}$. Since every vertex in H_i has degree at least two (in the undirected sense) this implies that $\sum_{d_{H_i}(u)>2}[d_{H_i}(u)-2]\le 2(c_i+\sqrt{2q_i})$. By the recursive definition of q_{i+1} we have $q_{i+1}-q_i = 2(2k-2+\sqrt{2q_i}) = 2(c_i+\sqrt{2q_i})$. Now we see that (A) holds for $i+1$. To see that (B) holds given (A) it suffices to observe that every vertex has degree at least 2 in every H_j constructed so far and at most 2 in each P_t, $t=1,2,\ldots,k$. Hence every vertex u contributing to the sum in (B) contributes with at least the same amount to the sum in (A).

Note that every vertex in Z_{i+1} must contribute at least $10k-c_{i+1}$ to the sum in (B), implying that $q_{i+1}\ge|Z_{i+1}|(8k+2)$. As we have seen that $q_{i+1}<16k^2$ this implies that $|Z_{i+1}|<2k$, which was the first part of (C). The second part of (C) follows immediately from this and the definition of W_i.

In order to prove that (D) holds for $i+1$, we note that if x_{i+1} or y_{i+1} are used in any subgraph $H_j\in\{H_1,H_2,\ldots,H_{i-1}\}$, then it must be because it lies on the corresponding path, P_j, in which case it will have degree at most 2 in H_j. This implies that the degrees of x_{i+1} and y_{i+1} are at most 2 in each of the subgraphs $H_1,H_2,\ldots,H_{i-1},P_{i+1},P_{i+2},\ldots,P_k$, implying (D).

We have now constructed $H_1,H_2,\ldots,H_k$. Let $H^*=V(H_1)\cap V(H_2)\cap\ldots\cap V(H_k)$ and $W^*=X\cup Y\cup\{z|d_{R^*}(z)\ge 10k\}$, where $R^*=A(H_1)\cup A(H_2)\cup\ldots\cup A(H_k)$. Note that $V(T)-H^*\subseteq W^*$, as $W_i\subseteq W^*$. Let $w\in W^*$ be arbitrary, and Assume that $w\notin Z_i$ but $w\in Z_{i+1}$. By the construction of H_j, $j\ge i+1$ this means that w is incident to at most 2 arcs in each H_j. By the construction of H_i we have $|A(H_i)|\le|V(H_i)|+c_i+\sqrt{2q_i}$. This implies that $d_{H_i}(w)\le 2+2c_i+2\sqrt{2q_i}$ (consider any ear decomposition of H_i). Now we see that

$$\begin{aligned}
d_{R^*}(w) &\le d_{L_i}(w)+d_{H_i}(w)+\sum_{j=i+1}^{k} d_{H_j}(w)\\
&\le 10k+(2+2c_i+2\sqrt{2q_i})+2(k-i)\\
&\le 10k+(16k+2)+2k\\
&\le 28k+2\le 30k.
\end{aligned}$$

Furthermore $|W^*|\le 4k$ by a similar argument as when we proved (C) above (using that $q_{k+1}\le 16k^2$). As $\delta^0(T)\ge 37k$ this implies that every vertex in W^* has at least $3k$ arcs into H^* and $3k$ arcs from H^*. Therefore it is not difficult to connect every vertex in W^* to every H_i, which it does not already belong to, by disjoint arcs (one in each direction). This gives us the desired arc-disjoint spanning strong subdigraphs. ∎

6. Further Consequences and Open Problems

The following result, which proves a conjecture of Bang-Jensen and Gutin from [3] (see also [2, Conjecture 9.9.12.]), is an immediate consequence of Theorem 5.4. Note that if Conjecture 1.1 is true, then already $2k$-arc-strong connectivity suffices.

Theorem 6.1. *Let $T=(V,A)$ be a $74k$-arc-strong tournament and let $u_1,\dots,u_k$, $v_1,\dots,v_k$ be not necessarily distinct vertices of T. Then T contains $2k$ arc-disjoint branchings $F^+_{u_1},\dots,F^+_{u_k},F^-_{v_1},\dots,F^-_{v_k}$ such that $F^+_{u_i}$ is an out-branching rooted at u_i and $F^-_{v_i}$ is an in-branching rooted at v_i for $i=1,2,\dots,k$.*

Note also that we only proved Theorem 5.4 for tournaments. Our proof depends on the fact that $X\cap Y=\emptyset$ (with X,Y defined in the proof) and this is not always the case for semicomplete digraphs. However, since every semicomplete digraph D contains a spanning tournament with arc-connectivity at least $\lambda(D)/2$ (see e.g. [2, Theorem 7.14.1]) it follows that the conclusion of Theorem 5.4 (and Theorem 6.1) holds when we double the requirement on the degree (connectivity).

As we mentioned in the introduction, Conjecture 1.1 contains the Kelly conjecture as a special case. On the other hand, one can construct k-arc-strong tournaments T on arbitrarily many vertices for which $\lambda(T-x)<k$ for every $x\in V(T)$ (for instance by modifying slightly the idea used by Thomassen on page 166 of [10]). Hence Conjecture 1.1 does not seem to follow easily from the Kelly Conjecture. The following two Conjectures represent successive weakenings of Conjecture 1.1.

Conjecture 6.2. Let k,s and t be natural numbers such that $k=s+t$. Then every k-arc-strong tournament contains arc-disjoint spanning strong subdigraphs D_1,D_2 such that D_1 is s-arc-strong and D_2 is t-arc-strong.

Conjecture 6.3. Every k-arc-strong tournament contains a spanning strong subdigraph H such that $T-A(H)$ is $(k-1)$-arc-strong.

Thomassen proved [10, Theorem 4.2] that every 2-arc-strong tournament T contains a hamiltonian path P such that T–$A(P)$ is strong. It is interesting

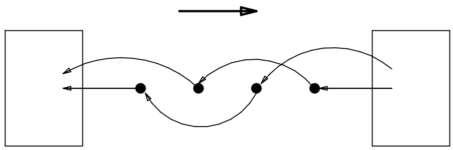

Figure 4. An infinite family of 2-arc-strong tournaments such that the deletion of the arcs of any hamiltonian cycle leaves a non-strong digraph. The first and the last box symbolizes arbitrary 2-arc-strong tournaments and the fat arc indicates that except for the 6 arcs shown from right to left all other arcs to from left to right.

to note that we cannot replace hamiltonian path by hamiltonian cycle above, as shown by the infinite class of 2-arc-strong tournaments in Figure 4.

Conjecture 6.4. Except for finitely many exceptions for each k, every k-arc-strong semicomplete digraph can be decomposed in k arc-disjoint spanning strong subdigraphs.

As pointed out by Thomassen in [10] there is no degree r of arc-strong connectivity which guaranties that every r-arc-strong tournament contains two arc-disjoint hamiltonian cycles. In fact, a tournament may have arbitrarily high arc-strong connectivity and still deleting a single arc may destroy all hamiltonian cycles. Thomassen also mentions a construction of Jackson showing that a tournament may have arbitrary high arc-strong connectivity without having 4 arc-disjoint hamiltonian paths. On the other hand Thomassen conjectures that there is some $\alpha(k)$ such that every $\alpha(k)$-strong tournament contains k arc-disjoint hamiltonian cycles. He shows that $\alpha(2) > 2$ and conjectures that every 3-strong tournament contains 2 arc-disjoint hamiltonian cycles. By a result of Fraisse and Thomassen [7] (saying that every k-strong tournament has a hamiltonian cycle which avoids any prescribed set of $k-1$ arcs), this conjecture would follow from the following conjecture.

Conjecture 6.5. Every tournament T either contains two arc-disjoint hamiltonian cycles or a set A' of at most two arcs such that $T - A'$ has no hamiltonian cycle.

As we mentioned in the introduction, for general digraphs almost nothing is known about decompositions into spanning strong subdigraphs and furthermore it is NP-complete to decide whether a given digraph has a decomposition into two strong spanning subdigraphs.

Theorem 6.6 (Yeo, unpublished manuscript). *It is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles.*

Corollary 6.7. *It is NP-complete to decide whether a digraph contains two arc-disjoint spanning subdigraphs.*

Conjecture 6.8. [9] There exists a natural number K such that every K-arc-strong digraph contains arc-disjoint branchings F_v^+, F_v^-, where F_v^+ is an out-branching rooted at v and F_v^- is an in-branching rooted at v.

We believe that a much stronger result holds:

Conjecture 6.9. There exists a natural number K such that every K-arc-strong digraph contains two arc-disjoint strong spanning subdigraphs.

We close with some remarks on decompositions of undirected graphs. Since every $4k$-edge-connected graph has $2k$ edge-disjoint spanning trees every $4k$-edge-connected graph G contains edge-disjoint spanning k-edge-connected spanning subgraphs H_1, H_2. There are many other ways to obtain a k-edge-connected graph than just taking the union of k edge-disjoint spanning trees. Since there exist 3-edge-connected graphs with no two edge-disjoint spanning trees $4k$ is best possible for $k=1$. But perhaps it can be improved when $k>1$.

Problem 6.10. Does there exist a constant c such that every $2k+c$-edge-connected graph contains two edge-disjoint k-edge-connected spanning subgraphs.

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